plane. This contention is supported by the present 38% reduction in dz_s/dx_s, compared with a 28% reduction in dy_s/dx_s.

Sokolov et al.[8] found that the coordinates $\xi_1 = x_s/U_1 t_s$ and $\eta_1 = y/\delta$, where δ is the thickness of the hypothetical turbulent boundary layer originating near the disturbance, provided a more adequate description of the evolution of the spot than the conical similarity coordinates.[9] It was suggested that the use of δ would allow the possible influence of Reynolds number and pressure gradient to be taken into account. Here, δ, whose precise determination is awkward, has been replaced by y_s, the more easily determined height of the spot. The contours shown in Fig. 4a using conical similarity coordinates show considerable and systematic variation, especially at the leading edge. By contrast, the contours in Fig. 4b represent a marked improvement, both for $m = 0$ and $m = 0.05$. Similarly, when the spot half-width z_s is used as the normalizing length scale, the use of coordinates ξ_1 and z/z_s showed a significant improvement over the conical coordinates. However, the collapse was not as good as in Fig. 4b, especially near the leading edge, probably because the (t_s, z) plane measurements were made at a constant y rather than a constant y/y_s. Another reason may be the greater sensitivity of the (t_s, z) plane data, relative to data in the (t_s, y) plane, to contamination from the side walls. We estimate this contamination to become important at $x \approx 2.7$ m in this experiment; the present kinematic scaling is unlikely to apply beyond this value of x.

Conclusions and Remarks

1) With reference to zero pressure gradient, a favorable pressure gradient reduces the rate of growth in all three spatial directions. For $m = 0.05$, the spanwise growth rate is reduced by nearly 38%, compared with decreases of about 35 and 28% in the longitudinal and main shear directions, respec tively.

2) The observed increase in U_{TE} for $m = 0.05$ is consistent with the suggestion that the favorable pressure gradient decreases the number of structures that make up the spot.

3) Conical similarity variables do not adequately describe the overall development of the spot. The data are, however, reasonably consistent with similarity when the local height and half-width of the spot are used.

Acknowledgment

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Radial Flow Between Two Closely Placed Flat Disks

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Introduction

THE flow within two closely placed flat disks can be included among those theoretical problems in fluid dynamics where, as the present paper will show, one can obtain a simple closed-form solution of the dependent variables that agrees well with the experiment. The practical aspect of the solution should not be overlooked, since disk-type devices are widely used in many engineering applications.[1] Several papers in the past dealt with the theoretical and/or experimental perspective of the problem. Kwok and Lee[2] collected and presented many experimental results. Due to the fact that velocity measurements in an extremely narrow gap (0.51 mm) are very difficult to obtain, only radial pressure distributions were presented. From the theoretical angle, most of the studies focused on integral solutions of the governing equations, with the aim to obtain the pressure variation while preserving the mean characteristics of the flow.[1-4] In a recent paper, Lee and Lin[5] attempted a differential solution, with the purpose once more to obtain the pressure distribution. Their results have shown good agreement with the experiment. However, since their linearized momentum equation cannot be solved exactly, the solution has been obtained numerically.

In the present paper, the flow within the gap is solved analytically. Simple equations for the radial static pressure distribution, the radial component of the velocity, and the global frictional coefficient are derived.

Analysis

Consider the steady, laminar, axisymmetric, nonswirling flow between the gap of two narrowly spaced flat disks shown in Fig. 1. Because the gap is small ($2h/R_o$ in the order of 1×10^{-3}), the axial velocity component of the velocity can be neglected. Under these assumptions, the equations of motion in nondimensional form are

Continuity:

$$\frac{\partial \bar{V}_r}{\partial \bar{r}} + \frac{\bar{V}_r}{\bar{r}} = 0 \tag{1}$$

R-momentum:

$$\bar{V}_r \frac{\partial \bar{V}_r}{\partial \bar{r}} = -\frac{d(\bar{\Delta} P)}{d\bar{r}} + \frac{1}{\bar{Re}}\left[\xi^{\zeta}\left(\frac{\partial^2 \bar{V}_r}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{V}_r}{\partial \bar{r}} - \frac{\bar{V}_r}{\bar{r}^2}\right) + \frac{\partial^2 \bar{V}_r}{\partial \bar{z}^2}\right] \tag{2}$$

where $\bar{V}_r = V_r/V_o$, $\bar{r} = \bar{r}/R_o$, $\bar{z} = z/h$, $\overline{\Delta P} = [P(r) - P_o]/\rho V_o^2$, $\xi = h/R_o$, $\bar{Re} = Re\xi$, $Re = \rho V_o h/\mu$, V_o is the inlet velocity, P_o is the inlet static pressure, ρ is the fluid density, and μ is the absolute viscosity.

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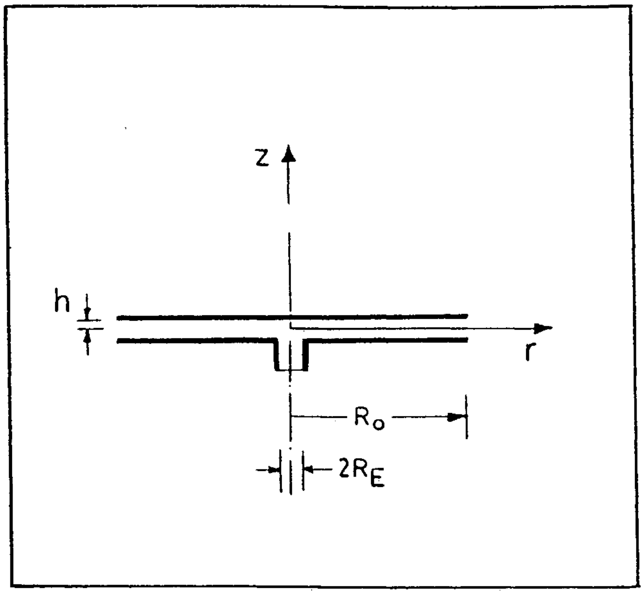

Fig. 1 Schematic of the physical problem.

As the fluid streams toward the center, the effective flow area decreases. The radial velocity must increase accordingly to satisfy continuity. A mathematical formulation that is to approximate well the real problem must include the inertia term in Eq. (2). This equation is nonlinear. Although it might be possible to integrate exactly Eq. (2) using Weierstrass or Jacobian elliptic functions, the solution will not be expressible in a closed form. In order to obtain a closed-form solution and avoid mathematical difficulties, the following assumption is made:

$$\bar V_r \frac{\partial \bar V_r}{\partial \bar r} \cong -\frac{1}{\bar r^3}$$

With the latter assumption, one considers only the average changes of the radial variations of the radial momentum. Then the linearized Eq. (2) is given by

$$-\frac{1}{\bar r^3} = -\frac{\mathrm{d}(\overline{\Delta P})}{\mathrm{d}\bar r} + \frac{1}{\bar{Re}}\left[\xi^{\xi}\left(\frac{\partial^2 \bar V_r}{\partial \bar r^2} + \frac{1}{\bar r}\frac{\partial \bar V_r}{\partial \bar r} - \frac{\bar V_r}{\bar r^2}\right) + \frac{\partial^2 \bar V_r}{\partial \bar z^2}\right] \tag{3}$$

If we now assume a solution of the radial velocity in the form

$$\bar V_r(\bar r, \bar z) = g(\bar r) f(\bar z) \tag{4}$$

then Eq. (1) yields

$$g(\bar r) = (a/\bar r) \tag{5}$$

where a is an arbitrary constant. From Eq. (4),

$$\bar V_r = (a/\bar r) f(\bar z) \tag{6}$$

Inserting Eq. (6) into Eq. (3) gives

$$\frac{\mathrm{d}(\bar\Delta P)}{\mathrm{d}\bar r} = \frac{1}{\bar r^3} + \frac{\lambda a}{\bar r} \tag{7}$$

and

$$\frac{\mathrm{d}^2 f(\bar z)}{\mathrm{d}\bar z^2} - \lambda \bar{Re} = 0 \tag{8}$$

where λ is the separation constant. Solution of Eq. (8) along with Eq. (6) yields

$$\bar V_r = (b\bar{Re}/2\bar r)(\bar z^2 - 1) \tag{9}$$

where $b = a\lambda$. The evaluation of the two constants of integration was made using the two boundary conditions:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\bar z}\right|_{\bar z = 0} = 0 \quad \text{and} \quad f(\bar z = 1) = 0$$

The arbitrary constant b in Eq. (9) is evaluated using continuity at the inlet:

$$b = -\frac{2}{\bar{Re}\int_0^1 (\bar z^2 - 1)\,\mathrm{d}\bar z} = \frac{3}{\bar{Re}} \tag{10}$$

Integration of Eq. (7) together with the boundary condition $\overline{\Delta P}(\bar r = 1.0) = 0$ gives

$$\overline{\Delta P}(\bar r) = \frac{6\bar r^2 \ell n \bar r - \bar{Re}(1 - \bar r^2)}{2\bar r^2 \bar{Re}} \tag{11}$$

If the assumption is made that the inertia terms in Eq. (2) dominate, then the pressure distribution will be given by

$$\overline{\Delta P}(\bar r) = [(\bar r^2 - 1)/2\bar r] \tag{12}$$

and the radial velocity by

$$\bar V_r = -(1/\bar r) \tag{13}$$

On the other hand, for dominant viscous effects (creeping flow),

$$\overline{\Delta P}(\bar r) = (3\ell n\bar r/\bar{Re}) \tag{14}$$

and

$$\bar V_r = (3/2\bar r)(\bar z^2 - 1) \tag{15}$$

A careful study of Eq. (11) will reveal that the radial pressure drop $(\overline{\Delta P})$ consists of the superposition of Eqs. (12) and (14). In other words, the total pressure drop along the radius must be partly due to acceleration of the fluid and partly due to viscous losses. The radial velocity distribution for the first case, where both inertia and viscous effects were present, and for the creeping flow is the same. The difference between the two flows comes from the pressure drop. If the same amount of fluid is to flow through the device, a higher pressure drop will be obtained in the former case than in the latter.

The shearing stresses at the wall can be calculated by

$$(\bar\tau_{r,z})_{\bar z = 1.0} = \frac{1}{\bar{Re}}\left.\frac{\partial \bar V_r}{\partial \bar z}\right|_{\bar z = 1.0} \tag{16}$$

where $\bar\tau_{r,z} = \tau_{r,z}/\rho V_o^2$. Then, from Eqs. (9) and (10), Eq. (16) yields

$$(\bar\tau_{r,z})_{\bar z = 1.0} = (3/\bar{Re}\bar r) \tag{17}$$

The local frictional coefficient C_f for the laminar flow between the plates is expressed by

$$C_f = 2(\bar\tau_{r,z})_{\bar z = 1.0}$$

From Eq. (17),

$$C_f' = (C_f \bar r/\xi) = (6/\bar{Re}) \tag{18}$$

According to Eq. (18), C_f' must be a function of the reduced Reynolds number $\bar{Re}$ alone. This conclusion will be validated using the experimental results of Ref. 4.

From Eq. (14), the pressure drop due to viscous effects is

$$[2(\overline{\Delta P})_{\text{vis}}/\ell n\bar r] = (6/\bar{Re}) \tag{19}$$

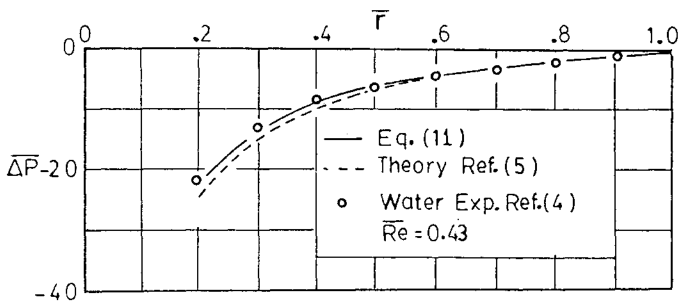

Fig. 2 Static pressure distribution.

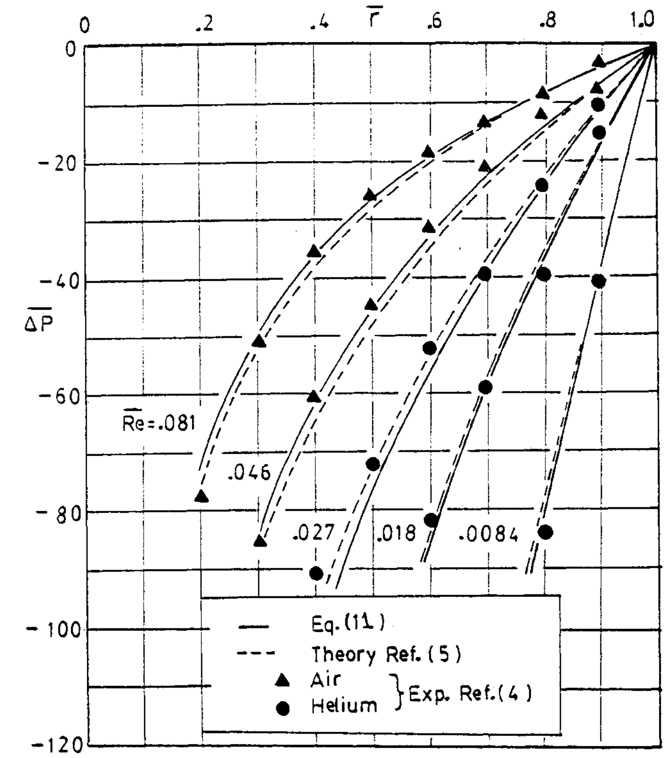

Fig. 3 Static pressure distribution.

However, from Eq. (18),

$$C_f' = [2(\overline{\Delta P})_{\text{vis}}/\ln \bar{r}] \tag{20}$$

From Eq. (11),

$$(\overline{\Delta P})_{\text{vis}} = \overline{\Delta P} + (1 - \bar{r}^2/2\bar{r}^2) \tag{21}$$

Inserting Eq. (21) into (20) yields

$$C_f' = \left| 2\overline{\Delta P} + \frac{(1-\bar{r}^2)}{\bar{r}^2} \right| \frac{1}{\ln \bar{r}} \tag{22}$$

Equation (22) will be used to obtain the value of C_f' from the experimental pressure distributions given by Hayes and Tucker.[4]

Results and Discussion

The results of the previously derived formulae are now presented. It is evident from Eq. (15) that the maximum velocity in every radial location is 1.5 times the local average. As the fluid flows radially toward the axis, the effective flow area decreases, and then radial velocity must increase hyperbolically to satisfy continuity. From Eqs. (11), (12), and (14), the pressure drop is seen to be the superposition of the other two. Physically, the total pressure gradient is due partly to azimuthal streamline convergence and partly to viscous dissipation. Increasing the viscosity of the fluid results in an increase of the viscous losses, with an equal increase in total pressure drop. The drop due to inertia will remain the same. For the limiting case where the reduced Reynolds number approaches zero ($\nu \rightarrow \infty$), $d\overline{\Delta P}/d\bar{r} \rightarrow \infty$ as the pressure attempts to deform a rigid body.

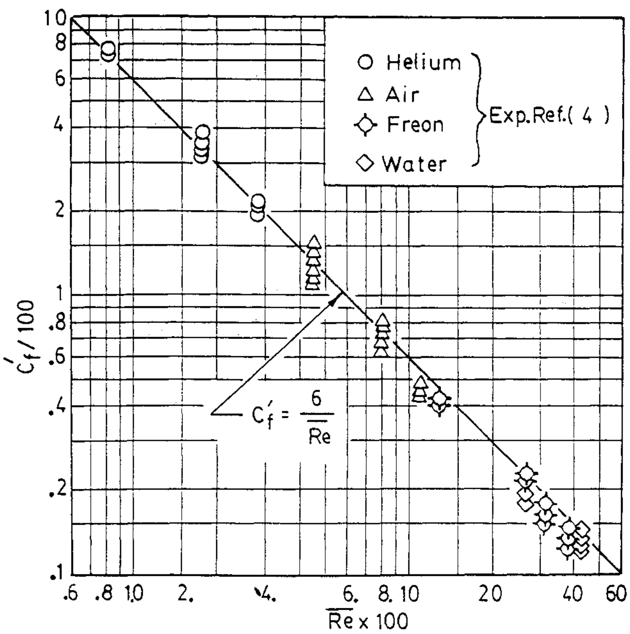

Fig. 4 Global frictional coefficient vs reduced Reynolds number.

A number of experiments were conducted by Hayes and Tucker.[4] Details of the experimental conditions are given by Lee and Lin.[5] Comparisons of the present theory with a sample of the experimental results of Ref. 4 and the analysis of Ref. 5 are presented in Figs. 2 and 3. In order to avoid overcrowding the figures, only the theoretical results of Lee and Lin[5] are used, since these are in better agreement with the experiments. It is evident that the present closed-form solutions agree better with the observations.

In Fig. 4, the global frictional coefficient C_f' is shown as a function of the reduced Reynolds number. A good agreement of Eq. (18) with the treated data of Ref. 4 is evident. Therefore, Eq. (18) represents a similarity relationship of the two dimensionless parameters C_f' and $\bar{R}_e$.

Acknowledgment

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